Make-Up Examination

Partial Differential Equations (MATH4220)
(Academic Year 2023/2024, Second Term)

Date: March 20th, 2024. **Time allowed:** 8:30-10:15

Recall that, the solution u for 1D heat equation

$$\begin{cases} \partial_t u = \partial_x^2 u, & \text{in } \ (t,x) \in [0,\infty) \times \mathbb{R}_x, \\ u_{|t=0} = \phi(x), & \text{for } \ x \in \mathbb{R}_x, \end{cases}$$

is given by

$$u(t,x) = \int_{\mathbb{R}} S(t,x-y)\phi(y)\mathrm{d}y, \quad \text{where } S(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

For any $r > 0$, we denote

$$B_r = \left\{ x \in \mathbb{R}^3 : |x| \le r \right\} \quad \text{and} \quad \partial B_r = \left\{ x \in \mathbb{R}^3 : |x| = r \right\}.$$

1. Consider the following four questions.
   (a) (5 points) State the definition of a linear PDE problem.
   (b) (5 points) State the definition of a harmonic function.
   (c) (5 points) State the definition of the well-posed problem.
   (d) (5 points) State the definition of the mean-value property.
   (e) (10 points) State the classification of the following second-order PDE:

   $$a_{11}\partial_x^2 u + 2a_{12}\partial_{xy}u + a_{22}\partial_y^2 u + a_1\partial_x u + a_2\partial_y u + a_0 u = 0,$$

   where $a_{11}^2 + a_{12}^2 + a_{22}^2 \neq 0$.

2. (15 points) Suppose that u is harmonic in $B_r \setminus \{0\}$ and satisfies

   $$u(x) = o(|x|^{-1}) \quad \text{as } x \to 0.$$

   Show that $u(x)$ can be defined at 0 and $u(x)$ is harmonic on B_r.

3. Let $u = u(x)$ be a regular function on $\mathbb{R}^3$.

(a) (5 points) Show that

$$\int_{\mathbb{R}^3} (x \cdot \nabla u)\Delta u \mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x.$$

(b) (5 points) Show that

$$\int_{\mathbb{R}^3} (x \cdot \nabla u) u \mathrm{d}x = -\frac{3}{2}\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x.$$

(c) (5 points) Show that

$$\int_{\mathbb{R}^3} (x \cdot \nabla u) u^3 \mathrm{d}x = -\frac{3}{4}\int_{\mathbb{R}^3} |u|^4 \mathrm{d}x.$$

4. (15 points) Let u be a regular solution of the following nonlinear elliptic equation,

$$-\Delta u + u - u^3 = 0, \quad \text{on } \mathbb{R}^3.$$

Show that

$$K_1(u) = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2 - |u|^4\right) \mathrm{d}x = 0,$$
$$K_2(u) = \int_{\mathbb{R}^3} \left(|\nabla u|^2 - \frac{3}{8}|u|^4\right) \mathrm{d}x = 0.$$

5. (10 points) Derive the formal solution formula to the problem

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), & \text{for } (t,x) \in (0,\infty)^2, \\ \partial_x u(t,0) = 0, & \text{for } t \in (0,\infty), \\ u(0,x) = \varphi(x), & \text{for } x \in (0,\infty), \end{cases}$$

by the method of reflection.

6. (a) (10 points) Let u be a regular function and satisfies

$$\partial_t u - \partial_x^2 u \le 0, \quad \text{on } \Omega = [0,T] \times [0,L].$$

Show that

$$\max_{\Omega} u(t,x) = \max_{\partial\Omega} u(t,x),$$

where $\partial\Omega = \{(t,x) \in \Omega |\text{ either } t = 0, \text{ or } x = 0, \text{ or } x = L\}$.

(b) (5 points) Denote $\Omega = [0,T] \times [0,L]$. Let u be a regular solution of

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x) + f(t,x), & \text{on } \Omega, \\ u(0,x) = 0, & \text{for } x \in [0,L], \\ u(t,0) = u(t,L) = 0, & \text{for } t \in [0,T]. \end{cases}$$

Show that

$$\max_{x \in L} u(t,x) \le t \max_{\Omega} |f(t,x)|, \quad \text{for any } t \in [0,T].$$

*************** END OF THE QUESTIONS ***************